LESSON 2: Endless Forms Most Beautiful

Key Characteristics of Quadratic Functions

Warm Up
Determine the slope and y-intercept of each linear function.

1. \( h(x) = 3x \)
2. \( g(x) = \frac{1}{2}(x - 5) \)
3. \( k(x) = x - 2 \)
4. \( m(x) = \frac{8x}{4} + 1 \)

Learning Goals
- Identify the factored form and general form of an equation for a quadratic function.
- Determine the equation for the axis of symmetry of a quadratic function, given the equation in general form or factored form.
- Determine the absolute minimum or absolute maximum point on the graph of a quadratic function and identify this point as the vertex.
- Describe intervals of increase and decrease in relation to the axis of symmetry on the graph of a quadratic function.
- Use key characteristics of the graph of a quadratic function to write an equation in factored form.

Key Terms
- second differences
- concave up
- concave down
- general form of a quadratic function
- factored form
- vertex of a parabola
- axis of symmetry

You have identified key characteristics of linear and exponential functions. What are the key characteristics of quadratic functions?
Dogs, Handshakes, Pumpkins, Ghosts

Consider the four quadratic models you investigated in the previous lesson. There are multiple equivalent ways to write the equation to represent each situation and a unique parabola to represent the equivalent equations. You can also represent the function using a table of values.

**Area of Dog Enclosure**

\[ A(s) = -2s^2 + 100s = -2(s)(s - 50) \]

<table>
<thead>
<tr>
<th>( s )</th>
<th>( A(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>98</td>
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<tr>
<td>2</td>
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<tr>
<td>3</td>
<td>282</td>
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<tr>
<td>4</td>
<td>368</td>
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</tbody>
</table>

**Handshake Problem**

\[ f(n) = \frac{1}{2}n^2 - \frac{1}{2}n = \frac{1}{2}(n)(n - 1) \]

<table>
<thead>
<tr>
<th>( n )</th>
<th>( f(n) )</th>
</tr>
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<tbody>
<tr>
<td>0</td>
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<tr>
<td>1</td>
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<td>4</td>
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</table>
Punkin’ Chunkin’

\[ h(t) = -16t^2 + 128t + 68 \]
\[ = -16\left(t - \frac{17}{2}\right)(t + \frac{1}{2}) \]

Ghost Tour

\[ r(x) = -10(x + 10)(x - 50) \]
\[ = -10x^2 + 400x + 5000 \]

1. Consider each representation.

a. How can you tell from the structure of the equation that it is quadratic?

b. What does the structure of the equation tell you about the shape and characteristics of the graph?

c. How can you tell from the shape of the graph that it is quadratic?

d. How can you tell from the table that the relationship is quadratic?
Let’s explore how a table of values can show that a function is quadratic. Consider the table of values represented by the basic quadratic function. This table represents the first differences between seven consecutive points.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>9</td>
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<tr>
<td>-2</td>
<td>4</td>
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<tr>
<td>-1</td>
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<td>4</td>
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<td>3</td>
<td>9</td>
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</table>

### First Differences

$4 - 9 = -5$
$1 - 4 = -3$
$0 - 1 = -1$
$1 - 0 = 1$
$4 - 1 = 3$
$9 - 4 = 5$

1. **What do the first differences tell you about the relationship of the table of values?**

Let’s consider the second differences. The second differences are the differences between consecutive values of the first differences.

2. **Calculate the second differences for $f(x)$. What do you notice?**
You know that with linear functions, the first differences are constant. For quadratic functions, the second differences are constant.

Let's consider the graph of the basic quadratic function, \( f(x) = x^2 \) and the distances represented by the first and second differences. Graph 1 shows the distances between consecutive values of \( f(x) \). The colored line segments are different lengths because the first differences are not the same.

Graph 1

Graph 2 shows the lengths of the first differences positioned along the \( x \)-axis. By comparing these lengths, you can see the second differences.

Think about:

Quadratic equations are polynomials with a degree of 2. Their second differences are constant. Linear functions are polynomials with a degree of 1, and their first differences are constant.
3. How does the representation in Graph 1 support the first differences calculated from the table of values?

4. How does the representation in Graph 2 support the second differences you calculated in the table?

5. Identify each equation as linear or quadratic. Complete the table to calculate the first and second differences. Then sketch the graph.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>First Differences</th>
<th>Second Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>-6</td>
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<td>-4</td>
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<table>
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<th>y</th>
<th>First Differences</th>
<th>Second Differences</th>
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</thead>
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<td>18</td>
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a. \( y = 2x \)  
b. \( y = 2x^2 \)
c. \( y = -x + 4 \)  

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>First Differences</th>
<th>Second Differences</th>
</tr>
</thead>
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<tr>
<td>-2</td>
<td>6</td>
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<td>-1</td>
<td>5</td>
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</tbody>
</table>

\[ x \quad y \quad x \quad y \quad x \quad y \quad x \quad y \]

\[ \begin{array}{cccc}
-3 & -5 & -3 & -5 \\
-2 & 0 & -2 & 0 \\
-1 & 3 & -1 & 3 \\
0 & 4 & 0 & 4 \\
1 & 3 & 1 & 3 \\
2 & 0 & 2 & 0 \\
3 & -5 & 3 & -5 \\
\end{array} \]

6. Compare the signs of the first and second differences for each function and its graph.

a. How do the signs of the first differences for a linear function relate to the graph either increasing or decreasing?

b. How do the signs of the second differences for quadratic functions relate to whether the parabola is opening upward or downward?

A graph that opens upward is identified as being concave up. A graph that opens downward is identified as begin concave down.
You know that different forms of an equation can reveal different characteristics about functions. Quadratic functions can be written in different forms.

A quadratic function written in the form \( f(x) = ax^2 + bx + c \), where \( a \neq 0 \), is in **general form**, or standard form. In this form, \( a \) and \( b \) are numerical coefficients and \( c \) is a constant.

A quadratic function written in **factored form** is in the form \( f(x) = a(x - r_1)(x - r_2) \), where \( a \neq 0 \).

1. Identify the general form and factored form of each equation in the Getting Started.

2. Consider the leading coefficient of each function equation in both general form and factored form.

   a. What does the leading coefficient tell you about the graph of each function?

   b. How is the leading coefficient related to the absolute minimum or absolute maximum of each function?

   c. How can you determine the \( y \)-intercept of the graph using general form?
3. Determine from the equation whether each quadratic function has an absolute maximum or absolute minimum. Explain how you know.

a. \( f(n) = 2n^2 + 3n - 1 \)  
b. \( g(x) = -2x^2 - 3x + 1 \)

c. \( r(x) = -\frac{1}{2}x^2 - 3x + 1 \)  
d. \( b(x) = -0.009(x + 50)(x - 250) \)

e. \( f(t) = \frac{1}{3}(x - 1)(x - 1) \)  
f. \( j(x) = 2x(1 - x) \)
The **vertex of a parabola** is the lowest or highest point on the graph of the quadratic function. The **axis of symmetry** or the line of symmetry of a parabola is the vertical line that passes through the vertex and divides the parabola into two mirror images. Because the axis of symmetry always divides the parabola into two mirror images, you can say that a parabola has reflectional symmetry.

**ACTIVITY 2.3**

**Axis of Symmetry**

1. Use patty paper to trace the graph representing the area of the dog enclosure. Then fold the graph to show the symmetry of the parabola and trace the axis of symmetry.
   
   a. Place the patty paper over the original graph. What is the equation of the axis of symmetry?
   
   b. Draw and label the axis of symmetry on the graph from your patty paper.

2. Analyze the symmetric points labeled on the graph.
   
   a. What do you notice about the y-coordinates of the points?
b. What do you notice about each point’s horizontal distance from the axis of symmetry?

c. How does the $x$-coordinate of each symmetric point compare to the $x$-coordinate of the vertex?

For a function in factored form, $f(x) = a(x - r_1)(x - r_2)$, the equation for the axis of symmetry is given by $x = \frac{r_1 + r_2}{2}$. For a quadratic function in general form, $f(x) = ax^2 + bx + c$, the equation for the axis of symmetry is $x = \frac{-b}{2a}$.

3. Identify the axis of symmetry of the graph of each situation from the Getting Started using the factored form of each equation.

4. Describe the meaning of the axis of symmetry in each situation, if possible.

5. Describe how you can use the axis of symmetry to determine the ordered pair location of the absolute maximum or absolute minimum of a quadratic function, given the equation for the function in factored form.

As you analyze a parabola from left to right, it will have either an interval of increase followed by an interval of decrease, or an interval of decrease followed by an interval of increase.

6. How does the absolute maximum or absolute minimum help you determine each interval?
Consider the graph of the quadratic function representing the Punkin’ Chunkin’ problem situation.

7. Determine the average rate of change between each pair. Then summarize what you notice.

a. points $A$ and $B$

b. points $A'$ and $B'$

c. points $B$ and $C$

d. points $B'$ and $C'$

e. What do you notice about the average rates of change between pairs of symmetric points?

The formula for the average rate of change is $\frac{f(b) - f(a)}{b - a}$. 
8. For each function shown, identify the domain, range, 
\(x\)-intercepts, \(y\)-intercept, axis of symmetry, vertex, and interval 
of increase and decrease.

a. The graph shown represents the function \(f(x) = -2x^2 + 4x\).

\[
\begin{array}{ll}
\text{Domain:} & \text{Range:} \\
\hline
\text{\(x\)-intercepts:} & \text{\(y\)-intercept:} \\
\hline
\text{Axis of symmetry:} & \text{Vertex:} \\
\hline
\text{Interval of increase:} & \text{Interval of decrease:}
\end{array}
\]

b. The graph shown represents the function 
\(f(x) = x^2 + 5x + 6\).

\[
\begin{array}{ll}
\text{Domain:} & \text{Range:} \\
\hline
\text{\(x\)-intercepts:} & \text{\(y\)-intercept:} \\
\hline
\text{Axis of symmetry:} & \text{Vertex:} \\
\hline
\text{Interval of increase:} & \text{Interval of decrease:}
\end{array}
\]
c. The graph shown represents the function \( f(x) = x^2 - x - 2 \).

\[
\begin{array}{c|c}
\text{Domain:} & \text{Range:} \\
\text{x-intercepts:} & \text{y-intercept:} \\
\text{Axis of symmetry:} & \text{Vertex:} \\
\text{Interval of increase:} & \text{Interval of decrease:}
\end{array}
\]

\[
\begin{array}{c|c}
\text{Domain:} & \text{Range:} \\
\text{x-intercepts:} & \text{y-intercept:} \\
\text{Axis of symmetry:} & \text{Vertex:} \\
\text{Interval of increase:} & \text{Interval of decrease:}
\end{array}
\]

d. The graph shown represents the function \( f(x) = x^2 - 3x + 2 \).

\[
\begin{array}{c|c}
\text{Domain:} & \text{Range:} \\
\text{x-intercepts:} & \text{y-intercept:} \\
\text{Axis of symmetry:} & \text{Vertex:} \\
\text{Interval of increase:} & \text{Interval of decrease:}
\end{array}
\]
You have analyzed quadratic functions and their equations. Let’s look at the factored form of a quadratic function in more detail.

1. A group of students each write a quadratic function in factored form to represent a parabola that opens downward and has zeros at \( x = 4 \) and \( x = -1 \).

   - **Maureen**
     My function is
     \[
     k(x) = -(x - 4)(x + 1).
     \]

   - **Tom**
     My function is
     \[
     g(x) = -2(x - 4)(x + 1).
     \]

   - **Tim**
     My function is
     \[
     m(x) = 2(x - 4)(x + 1).
     \]

   - **Micheal**
     My function is
     \[
     f(x) = -(x + 4)(x - 1).
     \]

   **a.** Sketch a graph of each student’s function and label key points. What are the similarities among all the graphs? What are the differences among the graphs?

   **b.** What would you tell Tim and Micheal to correct their functions?
c. How is it possible to have more than one correct function?

d. How many possible functions can represent the given characteristics? Explain your reasoning.

2. Consider a quadratic function written in factored form, \( f(x) = a(x - r_1)(x - r_2) \).

a. What does the sign of the \( a \)-value tell you about the graph?

b. What do \( r_1 \) and \( r_2 \) tell you about the graph?

3. Use the given information to write a function in factored form. Sketch a graph of each function and label key points, which include the vertex, the \( x \)- and \( y \)-intercepts.

a. The parabola opens upward, and the zeros are at \( x = 2 \) and \( x = 4 \).

b. The parabola opens downward, and the zeros at \( x = -3 \) and \( x = 1 \).
c. The parabola opens downward, and the zeros are at $x = 0$ and $x = 5$.

d. The parabola opens upward, and the zeros are at $x = -2.5$ and $x = 4.3$.

4. Compare your quadratic functions with your classmates’ functions. How does the $a$-value affect the shape of the graph?
5. For each quadratic function,

- Use the general form to determine the axis of symmetry, the absolute maximum or absolute minimum, and the $y$-intercept. Graph and label each characteristic.
- Use technology to identify the zeros. Label the zeros on the graph.
- Draw the parabola. Use the curve to write the function in factored form.
- Verify the function you wrote in factored form is equivalent to the given function in general form.

a. $h(x) = x^2 - 8x + 12$

zeros: ____________________

factored form: ________________
b. \( r(x) = -2x^2 + 6x + 20 \)

zeros: ________________

factored form: ________________

c. \( w(x) = -x^2 - 4x \)

zeros: ________________

factored form: ________________
d. $c(x) = 3x^2 - 3$

zeros: ________________

factored form: ________________
TALK the TALK

Quadratic Sleuthing

Use the given information to answer each question. Do not use technology. Show your work.

1. Determine the axis of symmetry of each parabola.
   a. The x-intercepts of the parabola are (1, 0) and (5, 0).
   b. The x-intercepts of the parabola are (−3.5, 0) and (4.1, 0).
   c. Two symmetric points on the parabola are (−7, 2) and (0, 2).

2. Describe how to determine the axis of symmetry given the x-intercepts of a parabola.

3. Determine the location of the vertex of each parabola.
   a. The function \( f(x) = x^2 + 4x + 3 \) has the axis of symmetry \( x = -2 \).
   b. The equation of the parabola is \( y = x^2 - 4 \), and the x-intercepts are (−2, 0) and (2, 0).

Think about:
Sketch a graph by hand if you need a model.
c. The function \( f(x) = x^2 + 6x - 5 \) has two symmetric points \((-1, -10)\) and \((-5, -10)\).

4. Describe how to determine the vertex of a parabola given the equation and the axis of symmetry.

5. Determine another point on each parabola.
   a. The axis of symmetry is \( x = 2 \), and a point on the parabola is \((0, 5)\).

   b. The vertex is \((0.5, 9)\), and an \( x \)-intercept is \((-2.5, 0)\).

   c. The vertex is \((-2, -8)\), and a point on the parabola is \((-1, -7)\).

6. Describe how to determine another point on a parabola if you are given one point and the axis of symmetry.
Assignment

Write
1. Describe the characteristics of a quadratic function that you can determine from its equation in general form.

2. Describe the characteristics of a quadratic function that you can determine from its equation in factored form.

Remember
The sign of the leading coefficient of a quadratic function in standard form or factored form describes whether the function has an absolute maximum or absolute minimum.

A parabola is a smooth curve with reflectional symmetry. The axis of symmetry contains the vertex of the graph of the function, which is located at the absolute minimum or absolute maximum of the function.

Practice
1. Analyze each quadratic function.

\[ g(x) = 12x - 4x^2 + 16 \quad h(x) = -\frac{1}{4}(x - 3)(x + 2) \]

   a. Identify the quadratic function as general form or factored form.
   b. Does the quadratic function have an absolute maximum or absolute minimum?
   c. Does the graph open upward or downward?
   d. Determine any intercepts from the given form of the function.

2. Analyze each quadratic function.

\[ f(x) = -\frac{2}{3}x^2 - 3x + 15 \quad g(x) = \frac{3}{4}x^2 + 12x - 27 \]

   a. Identify the axis of symmetry.
   b. Use the axis of symmetry to determine the ordered pair of the absolute maximum or absolute minimum value.
   c. Describe the intervals of increase and decrease.
   d. Sketch the graph based on the information you just calculated.
   e. Use technology to identify the zeros.
   f. Place two pairs of symmetric points on your graph. What is the average rate of change between these pairs of symmetric points?
   g. Write the function in factored form.

3. Given a parabola that opens downward and has zeros at \( x = -2 \) and \( x = 3 \).
   a. Represent it as a quadratic equation in factored form.
   b. Sketch a graph of the quadratic function.
   c. What is the axis of symmetry and \( y \)-intercept of the quadratic function?
Stretch
1. Sketch the graph $f(x) = -3x^2 - 4$. How could you change the quadratic function to make the graph open upward? Show the change on the graph.
2. How could you change the quadratic function $f(x) = -3x^2 - 4$ to shift the graph up or down? Show on the graph.
3. How could you change the quadratic function $f(x) = -3x^2 - 4$ to shift the graph right or left? Show the change on the graph.

Review
1. A camp wants to create a larger space for their albino rabbit, Clover. They want to reuse the materials from Clover’s current enclosure in the construction of the new enclosure. The perimeter of Clover’s current space is 6 feet. The perimeter of his new enclosure will be 3 times larger than his former enclosure.
   a. What is the area of the new enclosure $A(w)$ in terms of width, $w$?
   b. What is the maximum area of the new enclosure? What are the dimensions?
2. Is $7x^{2t} - 5x^{2t}$ equivalent to $35x^{2t}$? Justify your answer.
3. Is $(16^{3y})^6$ equivalent to $16^{18y}$? Justify your answer.
4. Alejandra has $900 to open a bank account. She wants to put her money in the bank where she will earn the most money over time. Alejandra has a choice between the Platinum Bank that offers an account with 3% compound interest and the Diamond Bank that offers an account with 4% simple interest.
   a. What is the function used to calculate the balance in each account based on the year, $t$? Describe each function.
   b. In which bank should Alejandra deposit her money? Explain your reasoning.